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Traveling wave modes of a plane layered anelastic earth

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Summary. Incorporation of attenuation into the normal mode sum representations of seismic signals is commonly effected by applying perturbation theory. This is fine for weak attenuation, but problematic for stronger attenuation. In this work modes of the anelastic medium are represented as complex superpositions of elastic eigenfunctions. For the P-SV system a generalized eigenvalue equation for the complex eigenwavenumbers and complex coefficients used to construct the anelastic eigenfunctions is derived. The generalized eigenvalue problem for the P-SV problem is exactly linear in the eigenwavenumber at the expense of doubling the dimension. The SH problem is exactly linear in the square of the eigenwavenumber. This is in contrast to a similar standing wave problem for the earth free oscillations (Tromp and Dahlen, 1990). Attenuation is commonly incorporated into synthetic seismogram calculations by introduction of complex frequency dependent elastic moduli. The moduli depend nonlinearly on the frequency. The independent variable in the standing wave free oscillation problem is the frequency, which makes the eigenvalue problem nonlinear. The choice of the wavenumber as the independent variable for the traveling wave problem leads to a linear problem. The Earth model may be transversely isotropic. Compressional waves, and both polarizations of shear waves (SV, SH) are treated.

Key words: Seismic attenuation; Seismic anisotropy; Surface waves and free oscillations; Theoretical seismology, Wave propagation

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1. Introduction

A surface-wave seismic or shallow-water ocean seismo-acoustic signal can be conveniently represented as a linear superposition of orthogonal modes for moderately low frequencies. Our definition of "moderately low" is model dependent, by which we mean that the model is a "few" wavelengths thick. A "few" is somewhat arbitrary, but may be as many as 100. The main point is that the frequency content of the signal is not high enough to justify using ray theory.

When modal representations of the seismic or seismo-acoustic wave-field have been applied to surface-waves of the solid Earth or shallow water sound propagation by the ocean acoustics community, it has been fairly common practice to include the effects of attenuation as a first order perturbation to modal eigenvalues (e.g. Ingenito, 1973; Aki and Richards, 1980; Zhou, 1985). First order perturbation theory ignores anelastic coupling between modes and requires that $k/(Q\delta k) \ll 1$ where k is the unperturbed wavenumber, δk is the unperturbed wavenumber spacing and Q is the spatial quality factor. Because at low frequencies a significant fraction of a shallow water seismo-acoustic or regional seismic signal may propagate in low Q bottom sediments, or low shear Q upper mantle $k/(Q\delta k)$ can be $O(1)$. This makes the use of perturbation theory invalid, and can introduce serious errors in mode sum acoustic signal synthesis. Ewing et al. (1992) report shear Q values in the range 20 - 50 for continental shelf sediments off the New Jersey. Lebedev and Nolet (2003) found shear Q as low as 40 in the upper mantle.

The severity of the error resulting from the improper treatment of attenuation in mode sum signal synthesis has been graphically illustrated by Day et al. (1989). Day et al. (1989) calculated synthetic seismograms for stratified models consisting of a high Q layer over a layered half space. The shear Q of the underlying half space was lower than the Q in the overlying layer, and half space shear speeds were lower than the compressional wave speed in the overlying layer. The signals calculated from modal summation dramatically overestimate the effect of the low shear Q on the complete signal. The mode summation results were compared with the results from a wavenumber

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integration routine (Apsel, R.J. and J.E. Luco, 1983), that is similar to SAFARI/OASES (Schmidt and Tango, 1986) in common use in ocean acoustics. Serious errors in the mode sum seismograms are traced by Day et al. (1989) to the way in which perturbation theory is applied to treat the anelastic problem. Specifically, the difficulty occurs with the use of the unperturbed elastic eigenfunctions in the anelastic problem. Because the problem is with the eigenfunctions themselves, it cannot be repaired with higher order perturbation corrections to the eigenvalues. The problem does not appear when using wavenumber integration routines, because the attenuation is incorporated directly through the use of complex, frequency dependent compressional and shear speeds.

If one wishes to retain the physical insight inherent in a modal representation of the wave-field, and properly incorporate the effects of anelasticity, one approach is to invoke the correspondence principle (e.g. Leitman and Fisher, 1973), and solve for the anelastic modes directly. The correspondence principle states that the equations of motion for a linear viscoelastic material are just the equations for a perfectly elastic material with the elastic moduli replaced with the complex, frequency dependent anelastic moduli. The anelastic moduli must be frequency dependent and satisfy the Kramers-Kronig relations to preserve causality.

The correspondence principle approach has been used by Yuen and Peltier (1982) and Buland et al. (1985) to model aspects of the free oscillations of the whole earth. To a limited extent, it has also been applied to shallow water propagation problems. Bucker and Morris (1965) employed the correspondence principle to solve for the anelastic eigenwavenumbers, and model the propagation loss for a shallow water problem with a fairly simple structure. It is standard procedure in wavenumber integration approaches to wave-field modeling, e.g. Apsel, R.J. and J.E. Luco, (1983), Schmidt and Tango (1986) where arbitrary attenuation is incorporated by introducing complex, frequency dependent elastic moduli.

There has been work on directly solving for the complex modes of a plane layered fluid-elastic medium. Ivansson and Karasalo (1992, 1993) and Ivansson (1997) have published a numerical algorithm based on the winding number theorem from complex analysis to directly search for the

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poles of the anelastic modes in the complex plane. The complex plane is tiled with boxes, and contour integrals are performed numerically around each box. They determine the number of poles within each box, and iteratively refine the search until each box contains a single pole. A final refinement is effected by switching to polar coordinates, and integrating around circular contours to more precisely isolate the anelastic mode eigenvalues.

We adopt a different approach in that we represent the anelastic modes as a complex superposition of elastic eigenfunctions. The effects due to anelastic mode coupling are explicitly included and there is no restriction on the magnitude of the damping. Our approach is a traveling wave adaptation of Tromp and Dahlen (1990), who derived an elegant solution for the free oscillations of an anelastic spherical earth in terms of the elastic eigenfunctions and eigenfrequencies.

We have derived a generalized eigenvalue equation for the complex eigenwavenumbers and complex coefficients used in the superposition of the elastic eigenfunctions to construct the anelastic eigenfunctions. Our generalized eigenvalue equation is strictly linear for the complex anelastic eigenwavenumbers. This is in contrast to the nonlinear eigenvalue equation for the anelastic eigenfrequencies of the free oscillations of the earth (Dahlen, 1981; Tromp and Dahlen, 1990). The reason for this difference is our choice of the frequency ω as the independent variable in the dispersion relation. Because of the standing wave nature of the earth free oscillation problem, ω is taken as the dependent variable in the dispersion relation. Since the anelastic moduli are frequency dependent, the eigenvalue equation for the anelastic free oscillations is nonlinear. Our derivation also includes the effects of vertical transverse isotropy, which has a single vertical axis of symmetry. A particular feature of transversely isotropic media is that the P-SV motion still decouples from the SH motion. This is not true for more general anisotropy. The case of transverse isotropy is particularly relevant for bottom-interacting shallow water sound propagation. Berge et al.(1991) felt that additional azimuthal anisotropy induced by ripples in the sediment surface or cracks would be very weak.

Bottom-interacting shallow water sound propagation requires special handling of the fluid-

solid interface in the P-SV problem. There are no inherent difficulties in treating this interface. The procedure is straightforward and discussed in the next section of the manuscript.

We include the SH problem for completeness, but discuss it only briefly. The derivation of the generalized eigenvalue equation for the anelastic eigenwavenumbers will be sketched.

2. Definitions and notation

We adopt the notational conventions of Takeuchi and Saito (1972), who treat seismic surface-waves and free oscillations explicitly for a transversely isotropic Earth. Our coordinate system is a right handed coordinate system with the propagation direction along the x axis, y positive into the paper, and z positive upward. As mentioned the P-SV (Rayleigh) motion decouples from the SH (Love) motion. The perfectly elastic displacements u_i and stresses σ_{ij} for P-SV are

$$
\begin{aligned}
u_x &= -iy_3(z;\omega,k)e^{i(\omega t - kx)} \\
u_y &= 0 \\
u_z &= y_1(z;\omega,k)e^{i(\omega t - kx)}
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
\sigma_{xx} &= \left(F\frac{dy_1}{dz} - kAy_3 \right) e^{i(\omega t - kx)} \\
\sigma_{yy} &= \left(F\frac{dy_1}{dz} - k(A - 2N)y_3 \right) e^{i(\omega t - kx)} \\
\sigma_{zz} &= y_2 e^{i(\omega t - kx)} \\
\sigma_{zx} &= -iy_4 e^{i(\omega t - kx)} \\
\sigma_{yz} &= \sigma_{xy} = 0.
\end{aligned}
\tag{2}
$$

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The boundary conditions on the y_i for P-SV are

$$y_i \quad (i = 1, 2, 3, 4) \quad \text{continuous}$$

$$y_2 = y_4 = 0 \quad \text{at the free surface} \quad z = 0 \tag{3}$$

$$y_i \quad (i = 1, 2, 3, 4) \quad \to 0 \quad \text{as} \quad z \to -\infty.$$

The displacements and stress for SH motion are

$$u_x = u_z = 0$$

$$u_y = y_1(z; \omega, k) e^{i(\omega t - kx)} \tag{4}$$

and

$$\sigma_{yz} = \left(L \frac{dy_1}{dz} \right) e^{i(\omega t - kx)}$$

$$\sigma_{xy} = -ik N y_1 e^{i(\omega t - kx)} \tag{5}$$

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{zx} = 0.$$

In addition, we introduce the definition for y_2 so that

$$\sigma_{yz} = y_2 e^{i(\omega t - kx)} = L \frac{dy_1}{dz} e^{i(\omega t - kx)} \tag{6}$$

The boundary conditions for the y_i for SH are

$$y_1, y_2 \quad \text{continuous}$$

$$y_2 = 0 \quad \text{at the free surface} \quad z = 0 \tag{7}$$

$$y_1, y_2 \to 0 \quad \text{as} \quad z \to -\infty.$$

ω is the real angular frequency, ρ is the density, k is the horizontal wavenumber for a perfectly elastic medium, and A, C, F, L and N are the five elastic moduli in Love's (1944) notation necessary

to characterize a transversely isotropic medium. When

$$A = C = \lambda + 2\mu \qquad L = N = \mu \qquad F = \lambda \tag{8}$$

the medium is isotropic.

Fluid layers and the boundary between fluid and solid layers require special treatment since fluids do not support shear. In the fluid

$$A = C = F = \lambda \qquad L = N = 0 \tag{9}$$

The displacements and stresses in the fluid are

$$
\begin{aligned}
u_x &= -i y_3(z;\omega,k) e^{i(\omega t - kx)} = i\frac{k}{\omega^2 \rho} y_2 e^{i(\omega t - kx)} \\
u_y &= 0 \\
u_z &= y_1(z;\omega,k) e^{i(\omega t - kx)}
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
\sigma_{xx} &= \lambda\left(\frac{dy_1}{dz} - ky_3\right) e^{i(\omega t - kx)} \\
\sigma_{zz} &= y_2 e^{i(\omega t - kx)} \\
\sigma_{zx} &= -i y_4 e^{i(\omega t - kx)} \\
\sigma_{yy} &= \sigma_{yz} = \sigma_{xy} = 0.
\end{aligned}
\tag{11}
$$

The boundary conditions on the y_i for fluids are

$$
\begin{aligned}
y_i \ \ (i = 1, 2) &\quad \text{continuous at the fluid} - \text{solid boundary} \\
y_3 &= -\frac{k}{\omega^2\rho} y_2 \quad \text{in the solid at the fluid} - \text{solid boundary} \\
y_4 &= 0 \quad \text{in the fluid and at the fluid} - \text{solid boundary in the solid .}
\end{aligned}
\tag{12}
$$

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In order to minimize the notational overhead, we make no distinction between the y_i's for the P-SV and SH problems. We will concentrate mainly on the P-SV problem. The meaning of the y_i's will be clear from context. The P-SV and SH problems are treated separately since they do not couple in transversely isotropic media with a vertical symmetry axis. The above equations of motion for a perfectly elastic medium may be represented in first order form as

$$\partial_z \mathbf{b}_{R,\mathrm{L}} = \mathbf{M}_{R,\mathrm{L}}\,\mathbf{b}_{R,\mathrm{L}}, \tag{13}$$

where the subscripts R (Rayleigh), L (Love) indicate whether we are referring to the P-SV, Eq. (1) - (3), or the SH, Eq. (4) - (7), systems of equations. The vectors $\mathbf{b}_{R,L}$ and the matrices $\mathbf{M}_{R,L}$ are defined for P-SV motion as

$$\mathbf{b}_{\mathrm{R}} = (y_1, y_2, y_3, y_4)^{\mathrm{T}} \tag{14}$$

and

$$\mathbf{M}_{\mathrm{R}} = \begin{bmatrix} 0 & \mathrm{C}^{-1} & k\mathrm{F}/\mathrm{C} & 0 \\ -\omega^2\rho & 0 & 0 & k \\ -k & 0 & 0 & \mathrm{L}^{-1} \\ 0 & -k\mathrm{F}/\mathrm{C} & \left[k^2\left(\mathrm{A} - \mathrm{F}^2/\mathrm{C}\right) - \omega^2\rho\right] & 0 \end{bmatrix}, \tag{15}$$

for SH motion as

$$\mathbf{b}_{\mathrm{L}} = (y_1, y_2)^{T} \tag{16}$$

and

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$$\mathbf{M}_{\mathrm{L}} = \begin{bmatrix} 0 & \mathrm{L}^{-1} \\ (k^2\mathrm{N} - \omega^2\rho) & 0 \end{bmatrix}, \tag{17}$$

and for fluids as

$$\mathbf{b}_{\mathrm{Fluid}} = (y_1, y_2)^T \tag{18}$$

and

$$\mathbf{M}_{\mathrm{Fluid}} = \begin{bmatrix} 0 & \left(\frac{1}{\lambda} - \frac{k^2}{\omega^2\rho} \right) \\ -\omega^2\rho & 0 \end{bmatrix}. \tag{19}$$

From this point on, we drop the subscripts R, L, and Fluid on the vector **b** and the matrix **M**. It will be apparent from context which system we mean. The following development will be for the P-SV system. Analogous results for the SH system are summarized at the end of the paper. The details of incorporating fluid-solid boundaries are well known. Excellent treatments are in Takeuchi and Saito (1972) and Aki and Richards (1980, pp280-281).

There are inherent symmetries in the equations of motion (Kennett et al., 1978 and Thomson et al., 1986) that can be exploited to construct compact expressions useful for very efficiently deriving the elastic wave dispersion relation, orthogonality relationships and other quantities. Define the matrices **R**, **S**, **Λ** and **Ξ** as

$$\mathbf{R} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix} \quad \text{and} \quad \mathbf{S} = \begin{bmatrix} \mathbf{\Xi} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Xi} \end{bmatrix}, \tag{20}$$

and

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{\Xi} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{21}$$

Using the matrices defined above, we can form various compositions of two stress-displacement fields. For example for P-SV, we can form

$$\partial_z(\mathbf{b}^T \mathbf{S} \mathbf{b}) = \mathbf{b}^T \left[\mathbf{M}^T \mathbf{S} + \mathbf{S} \mathbf{M} \right] \mathbf{b} \tag{22}$$

whereby we operate from the left and the right with the same stress displacement vector $\mathbf{b}$. The perfectly elastic modes are then $\mathbf{b}_m$ for $m = 0, 1, 2, ..., \infty$. The completeness of the $\mathbf{b}_m$ for the case of a plane layered medium with a free surface and a rigid lower boundary has been proved by Kirrmann(1995). The only approximation is that in any practical implementation, we employ a finite number of modes. When both sides are integrated with respect to z from $-\infty$ to 0, we obtain the dispersion relation for Rayleigh waves in a transversely isotropic medium (Aki and Richards, 1980).

$$\omega^2 I_1 = k^2 I_2 + k I_3 + I_4 \tag{23}$$

where

$$I_1 = \int_{-\infty}^{0} \rho(y_1^2 + y_3^2) \, dz \tag{24}$$

$$I_2 = \int_{-\infty}^{0} (L y_1^2 + A y_3^2) \, dz \tag{25}$$

$$I_3 = 2 \int_{-\infty}^{0} \left(L y_1 \frac{dy_3}{dz} - F y_3 \frac{dy_1}{dz} \right) dz \tag{26}$$

$$I_4 = \int_{-\infty}^{0} \left(C \frac{dy_1}{dz}^2 + L \frac{dy_3}{dz}^2 \right) dz. \tag{27}$$

The left hand side of Eq.(22) is a perfect differential in z, and after integration it vanishes when the boundary conditions are applied.

3. Derivation of the generalized eigenvalue equation

The complex generalized eigenvalue equation for the complex eigenwavenumbers is derived in a straightforward manner. Invoking the correspondence principle, we represent the equations of motion for an anelastic transversely isotropic medium as

$$\partial_z \mathbf{c} = \tilde{\mathbf{M}} \mathbf{c} \tag{28}$$

where $\mathbf{c}$ and $\tilde{\mathbf{M}}$ represent the stress-displacement vector and wave operator matrix, respectively for anelastic media. For our definition of $\tilde{\mathbf{M}}$, we take

$$\tilde{\mathbf{M}} = \begin{bmatrix} 0 & \tilde{C}^{-1} & \kappa\tilde{F}/\tilde{C} & 0 \\ -\sigma^2\rho & 0 & 0 & \kappa \\ -\kappa & 0 & 0 & \tilde{L}^{-1} \\ 0 & -\kappa\tilde{F}/\tilde{C} & \left[\kappa^2\left(\tilde{A} - \tilde{F}^2/\tilde{C}\right) - \sigma^2\rho\right] & 0 \end{bmatrix}. \tag{29}$$

The symbols $\tilde{A}$, $\tilde{C}$, $\tilde{F}$, $\tilde{L}$ and $\tilde{N}$ are the five complex frequency dependent elastic moduli for an anelastic transversely isotropic solid; κ is the eigenwavenumber for the anelastic solid; σ is the frequency, which we take to be real for propagating waves.

We take

$$\mathbf{c} = \mathbf{c}_n \tag{30}$$

where $\mathbf{c}_n$ is an eigenfunction of the anelastic medium and is a solution of Eq. (28). In addition, we

represent $\mathbf{c}_n$ as a complex linear superposition of the eigenfunctions of the perfectly elastic problem

$$\mathbf{c}_n = \sum_{m=0}^{\infty} Q_{mn}\mathbf{b}_m. \tag{31}$$

Q_{mn} is the matrix of complex coefficients that transforms the elastic eigenfunctions $\mathbf{b}_m$ to the anelastic solutions.

Employing the matrix $\mathbf{R}$ defined above, we form the composition of an anelastic mode $\mathbf{c}_n$ as represented by Eq. (31) and an elastic mode $\mathbf{b}_n$

$$\partial_z(\mathbf{b}_n^T \mathbf{R} \mathbf{c}_n) = \mathbf{b}_n^T \left[\mathbf{M}^T \mathbf{R} + \mathbf{R} \tilde{\mathbf{M}} \right] \mathbf{c}_n, \tag{32}$$

and integrate over z from $-\infty$ to 0. The elastic and anelastic problems satisfy the same boundary conditions, so the left hand side of Eq. (32) vanishes after the integration. By assuming that the elastic eigenfunction $\mathbf{b}_n$ and the anelastic eigenfunction $\mathbf{c}_n$ have the same real frequency so that

$$\omega^2 = \sigma^2, \tag{33}$$

we arrive at the following infinite generalized quadratic eigenvalue equation for the anelastic eigen-wavenumbers κ_n

$$\boxed{\mathbf{A}\mathbf{q}_n + \kappa_n \mathbf{B}\mathbf{q}_n + \kappa_n^2 \mathbf{C}\mathbf{q}_n = 0} \tag{34}$$

where

$$\mathbf{A} = -\int_{-\infty}^{0} \left\{ k_n^2 \left(\mathrm{A} - \mathrm{F}^2/\mathrm{C} \right) y_3^{(n)} y_3^{(m)} \right\}$$

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$$+ k_n \left[\left(y_1{}^{(n)} y_4{}^{(m)} + y_4{}^{(n)} y_1{}^{(m)} \right) - \frac{F}{C} \left(y_2{}^{(n)} y_3{}^{(m)} + y_3{}^{(n)} y_2{}^{(m)} \right) \right],$$

$$- \left(C^{-1} - \tilde{C}^{-1} \right) y_2{}^{(n)} y_2{}^{(m)} - \left(L^{-1} - \tilde{L}^{-1} \right) y_4{}^{(n)} y_4{}^{(m)} \Big\} \, dz, \tag{35}$$

$$\mathbf{B} = \int_{-\infty}^{0} \left[\left(y_1{}^{(n)} y_4{}^{(m)} + y_4{}^{(n)} y_1{}^{(m)} \right) - \frac{\tilde{F}}{\tilde{C}} \left(y_2{}^{(n)} y_3{}^{(m)} + y_3{}^{(n)} y_2{}^{(m)} \right) \right] dz, \tag{36}$$

$$\mathbf{C} = \int_{-\infty}^{0} \left[\left(\tilde{A} - \tilde{F}^2 / \tilde{C} \right) y_3{}^{(n)} y_3{}^{(m)} \right] dz. \tag{37}$$

As Eq. (38) shows, the eigenvectors $\mathbf{q}_n$ are the columns of the matrix $\mathbf{Q}$

$$\mathbf{Q} = (\ldots, \mathbf{q}_n, \ldots). \tag{38}$$

By making the assignment

$$\mathbf{I} \mathbf{r}_n = \kappa_n \mathbf{I} \mathbf{q}_n, \tag{39}$$

the quadratic generalized eigenvalue problem Eq. (34) can be converted to a linear generalized eigenvalue problem (Garbow et al., pp.49-50, 1977) at the expense of doubling the dimension of the system

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{A} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{q}_n \\ \mathbf{r}_n \end{bmatrix} = \kappa_n \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{q}_n \\ \mathbf{r}_n \end{bmatrix}. \tag{40}$$

Equation (40) is the main result of this note. The solution of this linear generalized matrix eigenvalue problem yields the complex eigenwavenumbers κ_n for the modes of an anelastic transversely isotropic medium and the eigenvectors $\mathbf{q}_n$. As Eq.(38) shows, the eigenvectors $\mathbf{q}_n$ of Eq. (40) are the columns of the transformation matrix Q_{mn} used to construct the anelastic eigenfunctions from the elastic eigenfunctions from Eq. (31). The linearity of Eq. (40) is an important point and should be contrasted with the result derived by Tromp and Dahlen (1990) for the free oscillations of the

earth. Their equation (3.5) is

$$\left[\boldsymbol{\Omega}^2 + \mathbf{V}\left(\sigma_k\right)\right]\mathbf{q}_k = \sigma_k^2\mathbf{q}_k. \tag{41}$$

The matrix $\boldsymbol{\Omega}$ is a diagonal matrix of eigenfrequencies for the perfectly elastic Earth; $\mathbf{q}_k$ is the kth column of the transformation matrix $\mathbf{Q}$; σ_k is the complex eigenfrequency of the kth anelastic mode; and $\mathbf{V}(\sigma)$ is an anelastic potential energy matrix that is a functional of products of the elastic eigenfunctions and the complex frequency dependent elastic moduli. The lateral standing-wave nature of the earth free oscillation problem leads to the choice of the frequency as the dependent variable. Because the elastic moduli depend nonlinearly on frequency, the problem, Eq. (41), for the complex eigenfrequencies and eigenvectors is nonlinear. The choice of wavenumber as the independent variable for the traveling wave problem leads to the linear problem we have derived above, Eq.(40). The nonlinear frequency dependence is contained in the elements of the $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ matrices of Eq.(29), Eq(30), and Eq.(31), respectively.

We have also derived similar results for the SH problem. We form the composition

$$\partial_z(\mathbf{b}_n^T \boldsymbol{\Xi} \mathbf{c}_n) = \mathbf{b}_n^T \left[\mathbf{M}^T \boldsymbol{\Xi} + \boldsymbol{\Xi} \tilde{\mathbf{M}}\right] \mathbf{c}_n. \tag{42}$$

As definitions of $\mathbf{b}$ and $\mathbf{M}$, we take Eqs. (16) and (30) for SH motion. For $\tilde{\mathbf{M}}$ we take Eq. (17) with k replaced by κ, and N and L replaced by $\tilde{N}$ and $\tilde{L}$, respectively. Likewise the definition of $\mathbf{c}$ follows from Eq. (30) and (31). Upon integrating Eq. (42) with respect to z from 0 to $-\infty$, carrying out some additional algebra, and again setting $\omega^2 = \sigma^2$, we arrive at

$$\boxed{\left(k_n^2\mathbf{I} - \kappa_n^2\mathbf{N} - \mathbf{L}\right)\mathbf{q}_n = \kappa_n^2\,\mathbf{q}_n} \tag{43}$$

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where

$$\mathbf{N} = \left[\int_{-\infty}^{0} N y_1^{(n)2} dz \right]^{-1} \int_{-\infty}^{0} \delta N y_1^{(n)} y_1^{(m)} dz \tag{44}$$

and

$$\mathbf{L} = \left[\int_{-\infty}^{0} N y_1^{(n)2} dz \right]^{-1} \int_{-\infty}^{0} \delta L \frac{dy_1^{(n)}}{dz} \frac{dy_1^{(m)}}{dz} dz. \tag{45}$$

The two terms δL and δN are the complex frequency dependent parts of the two shear moduli $\tilde{L}$ and $\tilde{N}$. The anelasticity tensor $\tilde{c}_{ijkl}$ for a linear viscoelastic material can be written

$$\tilde{c}_{ijkl} = c_{ijkl} + \delta c_{ijkl}(\omega). \tag{46}$$

Substituting the appropriate expressions from Eq.(46) for $\tilde{L}$ and $\tilde{N}$, we were able to separate the perfectly elastic part from the frequency dependent anelastic part for the SH problem. There is no restriction on the magnitude of δL and δN, making it possible to simplify Eqs.(38)-(39). It is possible to also write the complex moduli $\tilde{A}$, $\tilde{C}$, and $\tilde{F}$ as well for the P-SV problem, Eqs.(29)-(31), but this leads to algebraic complexity that is not particularly illuminating. So this has not been done for the P-SV problem. Also note that Eq. (43) is linear in κ_n^2. A final point is that both Eq. (40) and Eq. (43) are infinite eigenvalue equations. Any practical implementation will necessarily employ a truncated mode set.

4. Discussion and Conclusions

Advantages of using the elastic eigenfunctions as a basis for the anelastic eigenfunctions are: 1. The effect of anelasticity on individual modes can be examined in detail; 2. Although this paper is explicitly for laterally homogenous problems, the effect of range dependent attenuation could be studied by making the complex expansion coefficients range dependent. (Pannatoni (2011) has published a coupled mode solution for a range dependent all-fluid acoustic model, which

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includes mode coupling due to attenuation.) If the environment is not geometrically range dependent (laterally heterogeneous), we can employ the same elastic basis; 3. Used in conjunction with a general range dependent coupled mode program, the propagation physics of a strongly laterally heterogeneous dependent shallow water environment or regional Earth model can be studied in detail. We have the ability to isolate the influence of the geometry, and different aspects of the rheology of the medium on a propagating seismic or seismo-acoustic signal.

We derived equations Eq. (40) and Eq. (43) for the computation of anelastic surface-wave eigenfunctions in transversely isotropic media by expressing them in terms of a linear complex generalized eigenvalue equation for the P-SV system, and a linear eigenvalue equation for the SH system. No perturbation theory is needed. The completeness of the elastic modes for the laterally homogeneous Earth as a basis has been proved by Kirrmann (1995) for the case of a free surface and rigid lower boundary, which is the locked mode approximation. The anelastic modes are useful for modeling and characterizing seismo-acoustic signals propagating in a shallow water environment or regional seismic phases characterized by high attenuation and transverse isotropy. This is an environment where a perturbation treatment of the bottom or upper mantle properties applied to mode summation signal synthesis, which has been shown by previous authors to lead to erroneous results. The solution of equations (40) and (43) are used to represent the anelastic modes in terms of the elastic modes, permitting a detailed analysis of the physics of strongly bottom interacting acoustic propagation. The effects of transverse isotropy and attenuation, including attenuation induced dispersion, are properly accounted for. Causality is assured by making sure the complex, frequency dependent moduli satisfy the Kramers-Kronig relations. Stable, well-behaved numerical algorithms exist for solving the complex generalized eigenvalue problem, even in cases where the the matrices involved are near singular (Golub and Van Loan, 1989). The QZ method is suitable for the generalized eigenvalue problem. Our next step is the numerical implementation of Eq.(40) and Eq.(43). A suitable check would be a camparison with the results obtained from Ivansson and Karasalo(1992, 1993) and Ivansson's (1997) direct algorithm.

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14. ABSTRACT
Incorporation of attenuation into the normal mode sum representations of seismic signals is commonly effected by applying perturbation theory. This is fine for weak attenuation, but problematic for stronger attenuation. In this work modes of the anelastic medium are represented as complex superpositions of elastic eigenfunctions. For the P-SV system a generalized eigenvalue equation for the complex eigenwavenumbers and complex coefficients used to construct the anelastic eigenfunctions is derived. The generalized eigenvalue problem for the P-SV problem is exactly linear in the eigenwavenumber at the expense of doubling the dimension. The SH problem is exactly linear in the square of the eigenwavenumber. This is in contrast to a similar standing wave problem for the earth free oscillations (Tromp and Dahlen,1990). Attenuation is commonly incorporated into synthetic seismogram calculations by introduction of complex frequency dependent elastic moduli. The moduli depend nonlinearly on the frequency. The independent variable in the standing wave free oscillation problem is the frequency, which makes the eigenvalue problem nonlinear. The choice of the wavenumber as the independent variable for the traveling wave problem leads to a linear problem.

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